Maths. 231

$\boxed{\frac{dy}{dx} - y = e^{kx}}$ $y(0)=0$

$y' - p(x)y = r(x)$

~~1st linear 1st order~~

$\mu = e^{\int p(x)dx}$ $e^{-x} = \frac{1}{x}$ $\boxed{e^{-x}}$

PART II: WORK OUT PROBLEMS

Solve each problem in this part in detail giving the necessary justification (each contains 4 points)

$e^{\int p(x)dx}$

$e^{\int(-1)dx}$

$F = e^{\int(-1)dx} =$

$F = e^{-x}$

1) Suppose that k is a constant and consider the initial value problem

$$y' - y = e^{kx} : y(0) = 0$$

a) Find the solution if $k \neq 1$

b) Find the solution if $k = 1$

c) Show that the solution in (b) is the limit of the solution in (a) as $k \to 1$

Solution.

(a) if $K \neq 1$ $y' - y = e^{kx}$ - It is Linear the form of $y' + P(x)y = r(x)$

then, the soln is

since, $P(x) = -1$, $r(x) = e^{kx}$, $h(x) = \int P(x)\,dx = \int -dx = -x$

$$y(x) = e^{-h(x)}\left(\int e^{h(x)} \cdot r(x)dx + c\right)$$

$$y(x) = e^{-(-x)}\left(\int e^{-x} \cdot e^{kx}dx + c\right)$$

$$y(x) = e^{x}\left[\int e^{(k-1)x}dx + c\right]$$

$$y(x) = e^{x}\left[\frac{e^{(k-1)x}}{k-1} + c\right]$$ → This is General Soln for fvn if $K \neq 1$

but $y(0) = 0$

$$y(0) = e^{0}\left[\frac{e^{(k-1)0}}{k-1} + c\right] = 0$$

$$1\left(\frac{1}{k-1} + c\right) = 0$$

$$\Rightarrow \quad c = -\frac{1}{k-1}$$

Therefor $y(x) = \frac{e^{x}}{k-1}\left[e^{(k-1)x} - 1\right]$ - is soln of (a)

(b) If $K = 1$

then, the soln is $y' - y = e^{x}$ is linear

since $P(x) = -1$, $r(x) = e^{x}$, $h(x) = -x$ b/c $h(x) = \int P(x)dx$

$$y(x) = e^{-h(x)}\left[\int e^{h(x)} \cdot r(x)\,dx + c\right]$$

$e^{-x} \cdot e^{x}dx + c$

$x+c$

$y(x) = e^{-(-x)}\left[\int e^{-x} \cdot e^{x} dx + c\right]$ since

$y(x) = e^{x}\left[\int 1 dx + c\right]$ $e^{-x} \cdot e^{x} = 1$

$y(x) = e^{x}(x + c)$

but $y(0) = 0$

$y(0) = e^{0}(0 + c) = 0$

$\Rightarrow c = 0$

There for $y(x) = \underline{xe^{x}}$ is soln of $\underline{b}$

(c)

$\lim_{k \to 1} \frac{e^{x}}{k-1}\left(e^{(k-1)x} - 1\right) = xe^{x}$

$\lim_{k \to 1} \frac{e^{x}\left(e^{(k-1)x} - 1\right)}{k-1} = \underline{xe^{x}}$

if $k = 1$ $e^{x}\left(\frac{e^{(k-1)x} - 1}{k-1}\right) = \frac{0}{0}$ is a form of $\frac{0}{0}$

by L'hospital rule

$\lim_{k \to 1} \frac{\left(e^{x}\left[e^{(k-1)x} - 1\right]\right)'}{(k-1)'} = \lim_{k \to 1} \frac{e^{x}\left[xe^{(k-1)x} - 0\right]}{1} = xe^{x}$

c = final value of (c)

$\lim_{k \to 1} e^{x}\left(xe^{(k-1)x}\right) = xe^{x}$

$e^{x} x e^{(1-1)x} = xe^{x}$

$xe^{x} e^{0} = xe^{x}$

$\underline{\underline{xe^{x} = xe^{x}}}$ since $e^{0} = 1$

There for soln in (b) is the limit of the soln in (a) as $k - 1$

2) A large tank is partially filled with 100 gallon of fluid in which 10 pounds of salt is dissolved. Brine containing 0.5 pound of salt per gallon is pumped in to the tank at a rate of 6 gallon per minute. The well mixed solution is then pumped out at a slower rate of 4 gallon per minute.

a) Set up an initial value problem describing the situation.

b) Find the amount of salt in the tank at any time.

Sol^n (a) Volume of water in tank at any time $= 100 + 2t$

Flow rate of entering into the tank $= 6$ gal/min

flow rate of leaving out from tank $= 4$ gal/min

concentration of salt pumped into tank $= 0.5 \frac{\text{pounds}}{\text{gallon}}$

concentration of salt pumped out from tank $= \frac{A(t)}{100+2t} = ?$

when $t=0$ con^n salt in the tank is 10 pounds of salt

$\Rightarrow A(0) = 10$ pounds

Volume of water in the tank = initial volume of water $+ (6-4)\frac{\text{gal}}{\text{min}} \cdot t$

$V = 100 + 2t$

FrE $= 6$ gal/min

FrL $= 4$ gal/min

con^n of salt pumped into tank $= 0.5 \frac{\text{pounds}}{\text{gal}}$

con^n of salt pumped out from tank $= \frac{A(t)}{100+2t}$

$A'(t) = \text{FrE}(\text{con}^n \text{ salt pumped into tank}) - \text{FrL} \cdot \frac{A(t)}{100+2t}$

$A' = 6 \text{ gal/min} \cdot 0.5 \text{ pound/gal} - 4 \text{ gal/min} \cdot \frac{A}{100+2t}$

$A' = 3 - \frac{2A}{50+t}$

$A' + \frac{2A}{50+t} = 3$

$\therefore \boxed{A' + \frac{2}{50+t}A = 3}, \quad A(0) = 10$

(b) To find amount of salt in tank at any time is given by

$A' + \frac{2}{50+t}A = 3 \quad A(0) = 10$

This eq^n is linear differential eq^n a form of

$A' + P(t)A = r(t)$

$P(t) = \frac{2}{50+t}$

$r(t) = 3$

$h(t) = \int P(t)\,dt$

$= \int \frac{2}{50+t}\,dt$

$= 2\ln|50+t|$

A.J

$A' + \frac{2}{50+t}A = 3 \quad A(0) = 10$

$p(t) = \frac{2}{50+t}$

$r(t) = 3$

$h(t) = 2 Ln^{50+t}$

The soln is given by

$A(t) = e^{-h(t)}\left[\int e^{h(t)} \cdot r(t)\,dt + c\right]$

$= e^{-2Ln50+t}\left[\int e^{2Ln50+t} \cdot 3\,dt + c\right]$

$= e^{Ln\frac{1}{(50+t)^2}}\left[\int 3e^{Ln(50+t)^2}\,dt + c\right]$

$= \frac{1}{(50+t)^2}\left[\int 3(50+t)^2\,dt + c\right]$

$= \frac{1}{(50+t)^2}\left[\frac{3(50+t)^3}{3} + c\right]$

$A(t) = \frac{(50+t)^3}{(50+t)^2} + \frac{c}{(50+t)^2}$

$A(t) = (50+t) + \frac{c}{(50+t)^2} \quad A(0) = 10$

$A(0) = (50+0) + \frac{c}{(50+0)^2} = 10$

$50 + \frac{c}{(50)^2} = 10$

$\frac{c}{2500} = 10 - 50$

$c = -40 \cdot 2500$

$c = -100,000$

$\therefore \; A(t) = (50+t) - \frac{100,000}{(50+t)^2}$

$\boxed{A(t) = (50+t) - \frac{100,000}{(50+t)^2}}$ → is soln of b

3) Solve the system of differential equation

$$\begin{cases} y_1' = y_1 + 2y_2 + 2e^{4x} \\ y_2' = 2y_1 + y_2 + e^{4x} \end{cases} \begin{array}{l} \to \text{eqn. 1} \\ \to \text{eqn. 2} \end{array}$$

Solution..

$$y_1' = y_1 + 2y_2 + 2e^{4x}$$

$$y_1'' = y_1' + 2y_2' + 8e^{4x}$$

but in eqn-2 $\quad y_2' = 2y_1 + y_2 + e^{4x}$

then $\quad y_1'' = y_1' + 2(2y_1 + y_2 + e^{4x}) + 8e^{4x}$

$$y_1'' = y_1' + 4y_1 + 2y_2 + 2e^{4x} + 8e^{4x}$$

$$y_1'' = y_1' + 4y_1 + 2y_2 + 10e^{4x}$$

but from eqn-2 $\quad \boxed{\dfrac{y_1' - y_1 - 2e^{4x}}{2} = y_2} \to$ eqn-3

then $\quad y_1'' = y_1' + 4y_1 + 2\left(\dfrac{y_1' - y_1 - 2e^{4x}}{2}\right) + 10e^{4x}$

$$y_1'' = y_1' + 4y_1 + y_1' - y_1 - 2e^{4x} + 10e^{4x}$$

$$y_1'' = 2y_1' + 3y_1 + 8e^{4x}$$

$y_1'' - 2y_1' - 3y_1 = 8e^{4x}$ → this is Non-homogenous Linear system.

the soln is in the form of $y_1(x) = y_h(x) + y_p(x)$

corresponding. homogenous, $y_1'' - 2y_1' - 3y_1 = 0$

corresponding. characterstic, $r^2 - 2r - 3 = 0$

$$r = \frac{2 \pm \sqrt{4+12}}{2} = \frac{2 \pm 4}{2} = 3 \text{ or } -1 \qquad r_1 = 3,\ r_2 = -1$$

$$y_{1h}(x) = c_1 e^{3x} + c_2 e^{-x}$$

by Method of undetermined coefficients

$$y_{1p}(x) = Ae^{4x}$$

$$y_{1p}'(x) = 4Ae^{4x}$$

$$y_{1p}''(x) = 16Ae^{4x}$$

Since $r(x) = 8e^{4x}$ is form $ke^{\alpha x}$ It's trial solns is form of Ae^{4x}

→ see next page ←

→ substituting in the eq$\underline{n}$ $Y_1'' - 2Y_1' - 3Y_1 = 8e^{4x}$

since $Y_{1p}''(x) = 16Ae^{4x}$ $Y_{1p}'(x) = 4Ae^{4x}$ $Y_{1p}(x) = Ae^{4x}$

$\Rightarrow 16Ae^{4x} - 2(4Ae^{4x}) - 3(Ae^{4x}) = 8e^{4x}$

$(16 - 11)Ae^{4x} = 8e^{4x}$

$\frac{5Ae^{4x}}{e^{4x}} = \frac{8e^{4x}}{e^{4x}}$

$5A = 8 \Rightarrow A = \frac{8}{5}$

There for $Y_1 p(x) = \frac{8}{5}e^{4x}$

This implies $Y_1(x) = Y_1h(x) + Y_1p(x)$

$\boxed{Y_1(x) = c_1e^{3x} + c_2e^{-x} + \frac{8}{5}e^{4x}}$ → is sol$\underline{n}$ of $Y_1(x)$

for sol$\underline{n}$ $Y_2(x) = \frac{Y_1' - Y_1 - 2e^{4x}}{2}$ → from eq$\underline{n}$-3

but $Y_1'(x) = 3c_1e^{3x} - c_2e^{-x} + \frac{32}{5}e^{4x}$

Then, $Y_2(x) = \frac{3c_1e^{3x} - c_2e^{-x} + \frac{32}{5}e^{4x} - c_1e^{3x} - c_2e^{-x} - \frac{8}{5}e^{4x} - 2e^{4x}}{2}$

$\boxed{Y_2(x) = c_1e^{3x} - c_2e^{-x} + \frac{14}{5}e^{4x}}$ → is sol$\underline{n}$ of $Y_2(x)$

but $Y(x) = \begin{pmatrix} Y_1(x) \\ Y_2(x) \end{pmatrix}$

$= \begin{pmatrix} c_1e^{3x} + c_2e^{-x} + \frac{8}{5}e^{4x} \\ c_1e^{3x} - c_2e^{-x} + \frac{14}{5}e^{4x} \end{pmatrix}$

$$\boxed{Y(x) = c_1e^{3x}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2e^{-x}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{4x}\begin{pmatrix} \frac{8}{5} \\ \frac{14}{5} \end{pmatrix}}$$

This is sol$\underline{n}$ of system of differential equation.

4) A $10kg$ mass is attached to a spring, stretching it 0.7 meter from its natural length. The mass started motion from equilibrium position with an initial velocity of 1 meter per second in the upward direction. If the damping force constant due to air resistance is 90. Find the displacement of the mass at any time.

Sol^n $L = 0.7m$
$m = 10kg$
$V_0 = -1 m/s$ b/c it is upward direction
c = damping force constant $= 90$
$x_0 = 0m$ b/c it stretching from its natural length
$g = 9.8 m/s^2$

$\therefore m x'' + c x' + kx = 0$
Find $x(t) = ?$ but $mg = KL$
$K = \frac{mg}{L} = \frac{10 \times 9.8}{0.7} = \frac{98}{0.7} = 140$
$10x'' + 90x' + kx = 0$
$10x'' + 90x' + 140x = 0$ - divide by 10
$x'' + 9x' + 14x = 0$ → this is homogenous Eq^n
Corr. characteristic $r^2 + 9r + 14 = 0$
by quadratic formula $r = \frac{-9 \pm \sqrt{81-56}}{2}$
$r = \frac{-9 \pm 5}{2}$ $r_1 = \frac{-9+5}{2}$ $r_1 = -2$
$r_2 = \frac{-9-5}{2}$ $r_2 = -7$

$x_1(t) = c_1 e^{r_1 t} = c_1 e^{-2t}$
$x_2(t) = c_2 e^{r_2 t} = c_2 e^{-7t}$
$\therefore x(t) = c_1 e^{-2t} + c_2 e^{-7t}$ but we have initial condition
$x(0) = c_1 e^{-2(0)} + c_2 e^{-7(0)} = 0$
$c_1 + c_2 = 0$ → eq^n 1
$x'(t) = -2c_1 e^{-2t} - 7c_2 e^{-7t}$
$x'(0) = -2c_1 e^{-2(0)} - 7c_2 e^{-7(0)} = -1$
$x'(0) = -2c_1 - 7c_2 = -1$
$-2c_1 - 7c_2 = -1$ → eq^n 2

$V_0 = -1 m/s$ $\boxed{V_0 = x'}$
$x_0 = 0m$ $V_0 = x'$
$x'(0) = -1$ $x_0 = 0$ is often called initial condition
$x(0) = 0$

use simultaneous eq^n
$c_1 + c_2 = 0$ $c_1 = -c_2$
$-2(-c_2) - 7c_2 = -1$
$2c_2 - 7c_2 = -1$
$-5c_2 = -1$ $c_2 = \frac{1}{5}$
$c_1 = -\frac{1}{5}$

Therefor $\boxed{x(t) = -\frac{e^{-2t}}{5} + \frac{e^{-7t}}{5}}$

PART I: SHORT – ANSWER QUESTIONS

Read carefully and give the answer in its most simplified form in the space provided. Each blank space carries 2 points.

1. *Find the general solutions of the following ODEs.*

a) $\frac{dy}{dx} = 1 + x + y^2 + xy^2$.

Ans. $y(x) = \tan\left(\frac{x^2}{2} + x + c\right)$

b) $\frac{dy}{dx} = \frac{-x-2y}{y}, y \neq 0$.

Ans. $y(x) = \frac{x}{-\ln y + c} - x$.

c) $y'' + y' + 2y = 0$.

Ans. $y(x) = e^{-\frac{x}{2}}\left[c_1 \cos\frac{\sqrt{7}}{2}x + c_2 \sin\frac{\sqrt{7}}{2}x\right]$.

d)) $x^2y'' - 4xy' + 4y = x^3$.

Ans. $y(x) = c_1x^4 + c_2x - \frac{x^3}{2}$.

2. *The integrating factor that makes $(2xy^2 + 2xy)dx + (x^2y + x^2)dy = 0$ exact is*

Ans. $\frac{1}{y+1}$.

$\alpha = (a-1)$ $(1-a)$ $\frac{6-4}{\sqrt{45-0}}$

PART II: WORKOUT PROBLEMS:

Solve each problem in this part in detail giving all necessary justification in the space provided.

1. *Solve* $(y + y\cos(xy))dx + (x + x\cos(xy))dy = 0, y(\pi) = 1.$ **(4 Points)**

Sol$\underline{n}$, $M(x,y) = y + y\cos(xy)$ $\quad \frac{dM(x,y)}{dy} = 1 + \cos xy - \cancel{yx\sin xy}$

$N(x,y) = x + x\cos(xy)$ $\quad \frac{dN(x,y)}{dx} = 1 + \cos xy - yx\sin xy$

$\frac{\partial m}{\partial y} = 1 + \cos xy - yx\sin xy$ $\quad \Rightarrow \frac{\partial m}{\partial y} = \frac{\partial N}{\partial x}$

$\frac{\partial N}{\partial x} = 1 + \cos xy - yx\sin xy$

$\therefore$ It is exact

$F(x,y) = \int M(x,y)\,dx + K(y)$

$F(x,y) = \int (y + y\cos(xy))dx + K(y)$

$F(x,y) = xy + \sin xy + K(y)$

$\frac{\partial F(x,y)}{\partial y} = \frac{\partial}{\partial y} xy + \frac{\partial}{\partial y}\sin xy + \frac{\partial}{\partial y} K(y)$

$N(x,y) = x + x\cos xy + K'(y)$ $\quad$ since $\frac{\partial F(x,y)}{\partial y} = N(x,y)$

$\cancel{x + x\cos xy} = \cancel{x + x\cos xy} + K'(y)$

$0 = K'(y)$ $\quad$ * Do not confuse.

$K'(y) = 0$ $\quad$ the Integral of zero with respect to any variable is constant.

$\int K'(y)\,dy = \int 0\,dy$

$K(y) = C^*$ $\quad$ since that constant is unknown either zero or any real number.

Therefore

$F(x,y) = xy + \sin xy + C^* = C$

$\boxed{xy + \sin xy = C}$

When $x = \pi$, $y = 1$ $\quad C = ?$ $\quad$ since $\sin\pi = 0$

$\pi(1) + \sin\pi(1) = C$ $\quad \pi + \sin\pi = C$

$\Rightarrow \pi + 0 = C$ $\quad C = \pi$

Therefore $\boxed{xy + \sin xy = \pi}$ $\rightarrow$ is answer

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2. *Solve the differential equation* $x\frac{dy}{dx} + 6y = 3xy^{\frac{4}{3}}$.

(4 Points)

Sol$^{\underline{n}}$ $\frac{xy'}{x} + \frac{6y}{x} = \frac{3xy^{\frac{4}{3}}}{x}$

$y' + 6\frac{y}{x} = 3y^{\frac{4}{3}}$ ..... eq$^{\underline{n}}$ *

→ this is Equations reducible to linear form or Bernoulli Equations a form of $y' + P(x)y = r(x)$

$y' + \frac{6}{x}y = 3y^{\frac{4}{3}}$ $P(x) = \frac{6}{x}$ $r(x) = 3$ $\alpha = 4/3$

Let $v(x) = y_{(x)}^{(1-4/3)}$ $v(x) = y_{(x)}^{-1/3}$

$v'(x) = -\frac{1}{3}y^{-4/3}y'$ but $y' = 3y^{4/3} - \frac{6}{x}y$

$v'(x) = -\frac{1}{3}y^{-4/3}(3y^{4/3} - \frac{6}{x}y)$ ↳ from eq$^{\underline{n}}$ *

$v'(x) = -1 + \frac{2}{x}y^{-\frac{1}{3}}$ but $y^{-\frac{1}{3}} = v$

$v' = -1 + \frac{2}{x}v$

$v' - \frac{2}{x}v = -1$ — eq$^{\underline{n}}$ ** for this eq$^{\underline{n}}$ $P(x) = \frac{-2}{x}$

$r(x) = -1$

$v(x) = e^{-h(x)}\left[\int e^{h(x)}r(x)dx + c\right]$ $h(x) = \int P(x)\,dx$

$= e^{-(-\ln x)}\left[\int e^{\ln\frac{1}{x}} \cdot -1dx + c\right]$ $= \int -\frac{2}{x}dx$

$= -\ln x$

$= e^{\ln x}\left[\int e^{\ln\frac{1}{x}} \cdot (-dx) + c\right]$ since $e^{\ln x} = x$

$e^{\ln\frac{1}{x}} = \frac{1}{x}$

$= x\left[\int -\frac{1}{x}dx + c\right]$

$= x(-\ln x + c)$

$v(x) = -x\ln x + cx$ but $v(x) = y_{(x)}^{-\frac{1}{3}}$

$y_{(x)}^{-\frac{1}{3}} = -x\ln x + cx$

$y_{(x)}^{\frac{1}{3}} = \frac{1}{-x\ln x + cx}$

$$\boxed{y(x) = \frac{1}{(-x\ln x + cx)^3}}$$ → is a sol$^{\underline{n}}$ of differential equation

3. *Solve the differential equation* $y'' + y' - 2y = xe^x$. (4 Points)

Sol$\underline{n}$, ~~corresponding homogenous~~ $y''+y'-2y=0$
corresponding homogenous characterstic $\lambda^2+\lambda-2=0$

the soln is the form of $y(x) = y_h(x) + y_p(x)$

b/c It is Non-homogenous Linear system

$$\lambda^2+\lambda-2=0 \quad \lambda = \frac{-1 \pm \sqrt{1+8}}{2} \quad \lambda_1 = \frac{-1+3}{2} \quad \lambda_1 = 1$$

$$\lambda_2 = \frac{-1-3}{2} \quad \lambda_2 = -2$$

$$\therefore y_h(x) = c_1 e^x + c_2 e^{-2x} \qquad y_1 = e^x \quad y_2 = e^{-2x}$$

but the soln for particular or $y_p(x)$ is solved by Method of Variation of parameters

$$y_p(x) = -y_1 \int \frac{y_2 \cdot r(x)}{W(y_1, y_2)} dx + y_2 \int \frac{y_1 \cdot r(x)}{W(y_1, y_2)} dx$$

where $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$ since $y_1 = e^x$, $y_1' = e^x$, $y_2 = e^{-2x}$, $y_2' = -2e^{-2x}$, $r(x) = xe^x$

$$\Rightarrow W(y_1, y_2) = e^x(-2e^{-2x}) - e^{-2x}(e^x)$$
$$= -2e^{-x} - e^{-x}$$
$$= -3e^{-x}$$

$$\therefore y_p(x) = -e^x \int \frac{e^{-2x} \cdot xe^x}{-3e^{-x}} dx + e^{-2x} \int \frac{e^x \cdot xe^x}{-3e^{-x}} dx$$
$$= -e^x \int \frac{x\cancel{e^{-x}}}{-3\cancel{e^{-x}}} dx + e^{-2x} \int \frac{xe^x}{-3} dx$$
$$= -e^x \int -\frac{x}{3} dx + e^{-2x} \int -\frac{xe^x}{3} dx$$
$$= -e^x\left(-\frac{x^2}{6}\right) + e^{-2x}\left(-\frac{xe^x}{3} + \frac{1}{3}e^x\right)$$
$$\therefore y_p(x) = \frac{x^2 e^x}{6} - \frac{xe^{-x}}{3} + \frac{e^{-x}}{3}$$

by Integration by part.

$$\int -\frac{x}{3} e^x dx$$
$$u = -\frac{x}{3} \quad du = -\frac{1}{3}dx$$
$$dV = e^x dx \quad V = e^x$$
$$\int -\frac{x}{3} e^x dx = -\frac{x}{3}e^x - \int -\frac{1}{3} e^x dx$$
$$= -\frac{x}{3}e^x + \frac{1}{3}e^x$$

Therefor $y(x) = y_h(x) + y_p(x)$

$$\boxed{y(x) = c_1 e^x + c_2 e^{-2x} + \frac{x^2 e^x}{6} - \frac{xe^{-x}}{3} + \frac{e^{-x}}{3}} \rightarrow \text{Answer.}$$

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$\boxed{y = y_1c_1 + y_2c_2}$ (or $y_2 = y_1 v$ $\int$ $v = \int e^{-\int g(x)dx} dx$ $g(x) = \frac{2y_1' + r(x)y_1}{y_1}$

4 Given the DE, $y'' - \frac{2x}{1+x^2}y' + \frac{2}{1+x^2}y = 0$ and one of its solution $y_1(x) = x$. Find the second solution and write the general solution.

(4 Points)

Soln ⇒ This type of Equation is Reduction of order Homogeneous Equation,

It is a form of $y'' + p(x)y' + q(x)y = 0$

Since $p(x) = \frac{-2x}{1+x^2}$ $q(x) = \frac{2}{1+x^2}$ and $y_1(x) = x$

then the soln is. $y_2(x) = (y_1(x)) \int \frac{e^{-\int p(x)dx}}{(y_1(x))^2} dx$

$p(x) = -\frac{2x}{1+x^2}$

$-\int \frac{-2x}{1+x^2}dx = \int \frac{2x}{1+x^2}dx$ Let $v = 1+x^2$, $dv = 2xdx$

$= \int \frac{dv}{v} = \ln v$ but $v = 1+x^2$

$= \ln 1+x^2$

$y_1(x) = x$

$(y_1(x))^2 = x^2$

$y_2(x) = x\int \frac{e^{\ln 1+x^2}}{x^2} dx$

$y_2(x) = x\int \frac{1+x^2}{x^2}dx$

$y_2(x) = x\left(\int \frac{1}{x^2}dx + \int dx\right)$

$y_2(x) = x\left(-\frac{1}{x} + x\right)$

$\boxed{y_2(x) = x^2 - 1}$ → soln of $y_2(x)$ or Second Solution

$y(x) = y_1(x)c_1 + y_2(x).c_2$ since $y_1(x) = x$
$= c_1 y_1(x) + c_2 y_2(x)$. $y_2(x) = x^2 - 1$

$\boxed{y(x) = c_1x + c_2(x^2-1)}$ → is General Solution

5. *A large tank is filled to capacity with 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped into the tank at a rate of 5gal/min. The well mixed solution is pumped out at a faster rate of 10 gal/min..*

a) *Write out an initial value problem which when solved will tell you how many lb of salt are in the tank at any time t?* **(2 Points)**

soln V = volume of water in the tank = 500 − 5t

FrP_{in} = Flow rate of soln pumped into the tank = 5 gal/min

FrP_{out} = Flow rate of soln pumped out from the tank = 10 gal/min

con salt pumped into tank = 2 pounds/gallon

con salt pumped out from tank $= \frac{A(t)}{500-5t}$

amount of salt t=0 is zero.

$A'(t) = FrP_{in to}$ (con Salt pumped into) $- FrP_{out} \cdot \frac{A(t)}{500-5t}$

$A' = 5\text{ gal/min} \cdot 2\text{ pounds/gal} - 10\text{ gal/min} \cdot \frac{A}{500-5t}$

$A' = 10\text{ pounds/min} - \frac{2A}{100-t}$

$\boxed{A' + \frac{2A}{100-t} = 10}$, $A(0) = 0$ b/c it is pure water initially

b) *Find the amount of salt in the tank at any time?* **(2 Points)**

$A' + \frac{2A}{100-t} = 10 \quad A(0) = 0$

→ this is Linear differential eqn a form of $A' + p(t)A = r(t)$

where $p(t) = \frac{2}{100-t}$

$h(t) = \int p(t)dt = -2\ln 100-t$

$r(t) = 10$

The soln is given by

$A(t) = e^{-h(t)}\left[\int e^{h(t)} \cdot r(t)dt + c\right]$

$A(t) = e^{(-)-2\ln 100-t}\left[\int e^{-2\ln 100-t} \cdot 10\, dt + c\right]$

$A(t) = e^{\ln(100-t)^2}\left[\int 10 e^{\ln(\frac{1}{100-t})^2} dt + c\right]$

$A(t) = (100-t)^2\left[\int 10 \cdot \frac{1}{(100-t)^2} dt + c\right]$

since $e^{\ln(100-t)^2} = (100-t)^2$

$A(t) = (100-t)^2\left[\int \frac{10}{(100-t)^2} dt + c\right]$

$= (100-t)^2\left[10(100-t)^{-1} + c\right]$

$A(t) = (100-t)^2\left[\frac{10}{100-t} + c\right]$

→ see next page

$$A(t) = (100-t)^2\left[\frac{10}{100-t} + c\right]$$

but we have initial condition $A(0) = 0$

$$A(0) = (100-0)^2\left[\frac{10}{100-0} + c\right]$$

$$0 = 10{,}000\left[\frac{10}{100} + c\right]$$

$$0 = 10{,}000\left(\frac{1}{10} + c\right)$$

$$1000 + 10{,}000c = 0$$

$$10{,}000c = -1000$$

$$c = -\frac{1000}{10{,}000}$$

$$c = -\frac{1}{10}$$

$$A(t) = (100-t)^2\left[\frac{10}{100-t} - \frac{1}{10}\right]$$

$$A(t) = 10(100-t) - \frac{(100-t)^2}{10}$$

Therefore $\boxed{A(t) = 1000 - 10t - \frac{(100-t)^2}{10}}$

this Answer of b

6

Part I: Short Answer Items /Each Blank Worth's 2 points /

1. Given the differential equation $xy^2dx+(x^2y+y^3)dy=0$

   a) Is it homogenous or not? homogenous for degree zero.

   b) Find the general solution $\frac{xy^3}{3}-\frac{y^4}{12}+\frac{5y^4}{4}$

2. If $y=x$ is the first solution of the differential equation $x^2y''+xy'-y=0$, then the second linearly independent solution of the given differential equation is

   $y=1/x$ $\boxed{-\frac{1}{2x}}$

   $y_1=x$, $p(x)=1/x$, $q(x)=-1/x^2$

3. Let $(3x^2y-x^2)dx+dy=0$ be given

   $e^{\int\left(\frac{P_y-Q_x}{Q}\right)dx}$ $\int 3x^2\,dx$ e^{x^3} (10)

   a) Find the integrating factor μ of the equation e^{x^3}.

   b) Find its solution

$-\int\left(\frac{1}{x}\right)dx=\int\left(\frac{2u}{1+u^2}\right)du$

$-\ln|x|+\ln|c_1|=$

$(3e^{x^3}y-e^{x^3}x^2)dx+e^{x^3}dy$ $\quad e^{x^3}$ $\quad (e^{x^3})\cdot 3x^2e^{x^3}$

$y=x^u$ $(u=y/x)$

$y=x$, ...

$x^2y''+xy'-y=0$

assume $y=x^m$

the auxiliary form is

$\Rightarrow m^2+(1-1)m-1=0$

$\Rightarrow m^2-1=0$

$\Rightarrow m^2=1$

$\Rightarrow m=\pm1$

$\Rightarrow y_1=x,\ y_2=x^{-1}$, is the 2nd solution

$y_2=\frac{1}{x}$

$e^{x^3}-\frac{e^{x^3}}{3x^2}$

Good luck

Mid exam of math-231

Part II -Work Out

1. Find the general solution of the linear differential equation $\frac{dy}{dx} + xy = x$

$\boxed{y' + \cancel{a}y = x}$

$\frac{dy}{dx} + xy = x$

$\Rightarrow \frac{dy}{dx} = x - xy$

$\frac{dy}{dx} = x(1-y)$

$\frac{1}{1-y}\,dy = x\,dx$

integrating both side.

$\int \frac{1}{1-y}\,dy = \int x\,dx$

$-$

$\ln|1-y| = \frac{x^2}{2} + c$

$|1-y| = e^{x^2/2 + c}$

$1-y = \pm c\,e^{x^2/2}$

$-y = c\,e^{x^2/2} - 1$

$y = -c\,e^{x^2/2} + 1$

$y(x) = \cancel{\#}c\,e^{-x^2/2} + 1$

I.F $= e^{\int x\,dx}$

4

$y(x) =$

$y(x) =$

$y(x$

$y(x$

$y(x) =$

Mid exam of math-231

$d^2 + (m-1)d + b$ $\quad$ $4 - 4b < 0$

$d^2 + \quad m^2 + (m-1)a + b$

2. Given a differential equation $x^2y'' - 2xy' + 2y = xe^{-x}$

$y_H =$ Euler Equation $\quad m^2 + (m+2)a +$

$(m-1) - 2 + 2$

-a) Find the solution of the associated homogenous equation (2pts)

$-2m + 2 + 2$

$x^2y'' - 2xy' + 2y = xe^{-x}$ $\quad y'' - \frac{2}{x}y' + \frac{2}{x^2} = 0$ $\quad \boxed{m^2 - 2m +}$

$x^2y'' - 2xy' + 2y = 0$

$y = x^m$

Let $y = x^\lambda$, $y_1' = \lambda x^{\lambda-1}$, $y_1'' = \lambda(\lambda-1)x^{\lambda-2}$

$\Rightarrow x^2(\lambda(\lambda-1)x^{\lambda-2} - 2x(\lambda x^{\lambda-1}) + 2(x^\lambda)$

$\Rightarrow \left[x^2(\lambda^2-\lambda)\cdot\frac{1}{x^2} - 2x(\lambda\cdot\frac{1}{x}) + 2\right]\underbrace{x^\lambda}_{0} = 0$

$\left[\cancel{x^2}(\lambda^2-\lambda)\frac{1}{\cancel{x^2}} - 2\cancel{x}(\lambda\cdot\frac{1}{\cancel{x}}) + 2 = 0\right.$

$\Rightarrow \lambda^2 - \lambda - 2\lambda + 2 = 0$

$\Rightarrow \lambda^2 - 3\lambda + 2 = 0$

$\Rightarrow \lambda^2 - 2\lambda - \lambda + 2$

$\lambda(\lambda-2) - 1(\lambda-2) = 0$

$\lambda_1 = 1$ or $\lambda_2 = 2$

2

$y_h(x) = c_1x + c_2x^2$

~~Since $f(x) = xe^{-x}$ to find the particular~~

b) Find the particular solution (2pts)

$y_1 = x$ and $y_2 = x^2$

$W(y_1, y_2) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} \Rightarrow 0$

$2x^2 - x^2 \neq 0$

$x^2 \neq 0$

Then y_1 and y_2 are linearly independent.

Using Varation parameters.

$y_p(x) = -y_1(x) \int \frac{y_2(x)\, r(x)\, dx}{W} + y_2 \int \frac{y_1(x)\, r(x)\, dx}{W}$

$\Rightarrow -x \int \frac{x^2 \cdot x e^{-x} dx}{x^2} + x^2 \int \frac{x \cdot x e^{-x} dx}{x^2}$

$* \Rightarrow -x \int x e^{-x} dx + x^2 \int e^{-x} dx.$

$\int u \frac{dv}{dx} \cdot dx = u \cdot v - \int v \cdot \frac{du}{dx} \cdot dx$

Let $u = x \qquad \frac{du}{dx} = 1$

$\frac{dv}{dx} = e^{-x}$

$\int dv = \int e^{-x} dx$

$v = e^{-x}$

$\int u \frac{dv}{dx} \cdot dx = x \cdot (e^{-x}) - \int e^{-x} dx$

$x e^{-x} + e^{-x}$

$-x(x e^{-x} + e^{-x}) + x^2 \int e^{-x} dx$

$-x^2 e^{-x} - x e^{-x} + x^2 e^{-x} + c$

$y_p(x) = \cancel{(-x - 1 - x) x e^{-x}} + c$

$y_p(x) = \cancel{+x^2 e^{-x}} - x e^{-x} - x^2 e^{-x} + c$ (Ans)

2

c) Write the general solution of the given differential equation (1pt)

$y(x) = y_h(x) + y_p(x)$

$y(x) = \cancel{c_1 x + c_2 x^2 +}$

$y(x) = c_1 x + c_2 x^2 - x^2 e^{-x} - x e^{-x} - x^2 e^{-x} + c$

1

Good luck

-20-

3. Find the current at any time t in the RLC series circuit with the electromotive force of **160t** Volts , inductance of **0.1** Henry , resistance of **20** Ohms and capacitance of 10^{-3} Farads. **(5 pts)**

given

$E = 160t$ volts

$L = 0.1$ Henry

$R = 20$

$C = 10^{-3}$ Farad.

$RI + \frac{q}{c} = E(t)$

$R\frac{dq}{dt} = \frac{q}{c} = E(t)$

$\boxed{\frac{dq}{dt} + \frac{q}{Rc} = \frac{E(t)}{R}}$ its homogeneous,

5

$Lq'' + Rq' + \frac{1}{c}q = E$

$\frac{1}{10}q'' + 20q' + \frac{1}{\frac{1}{10^3}}q = 160t$

$\frac{1}{10}q'' + 20q' + 1000q = 160t$

$q'' + 200q' + 10^4 q = 1600t$

Find an integrated factor

$I.F = e^{\int \frac{1}{Rc} dt}$

$q(t) = \frac{\int \frac{E(t)}{R} \cdot I.F \, dt + c}{I.F}$

at homogeneous equation

$q'' + 200q' + 10^4 q = 0$

$\lambda^2 + 200\lambda + 10^4 = 0$

but $I(t) = \frac{dq}{dt}$

$\lambda = \frac{-200 \pm \sqrt{(200)^2 - 4(10^4)(0)}}{2}$

$\lambda = \frac{-200 \pm 0}{2} = -100$

$y_1 = e^{-100x}$ and $y_2 = e^{-100x}$

$y(x) = c_1 y_1 + c_2 y_2$

$y_h(x) = c_1 e^{-100x} + c_2 e^{-100x}$

since $r(x) = 1600t$ to Find

$r(x) = At^2 + Bt + c$

$r'(t) = 2At + B$

$r''(t) = 2A$

Good luck

$\lambda^2 + 200\lambda + 10^4 = 0$, $q'' + 200q' + 10^4 q = 1600t$

$r(t) = At^2 + Bt + c$

$r'(t) = 2At + B$

$r''(t) = 2A$

$2A + 200(2At + B) + 10^4(At^2 + Bt + c) = 1600t$

$\Rightarrow 2A + 400At + 200B + 10^4 At^2 + 10^4 Bt + 10^4 c = 1600t$

$2A + 200B + 10^4 c = 0$

$400A + 10^4 B = 1600$

$10^4 A = 0$

$A = 0$

$400(0) + 10^4 B = 1600$

$B = \frac{16\cancel{00}}{100\cancel{00}}$

$B = \underline{\underline{0.16}}$

$2(0) + 200(0.16) + 10^4 c = 0$

$10^4 c = -32$

$c = \underline{\underline{-32 \times 10^{-4}}}$

$r(t) = At^2 + Bt + c$

$r(t) = \underline{\underline{0.16t - 32 \times 10^{-4}}}$

$y(t) = y_h(x) + y_p(t)$

$y(t) = \underline{\underline{c_1 e^{-100x} + c_2 e^{-100x} + 0.16t - 32 \times 10^{-4}}}$ (Ans)

4. Suppose that the mass in a mass-spring dash pot system with m=1kg ,dashpot constant of $4\,kg/sec$ and spring constant of $4\,N/m$ set in motion under the influence of an external force $F(t)=10\cos 3t$ with $x(0)=0$ *and* $x'(0)=2$

a) Find the governing differential equation (2pts)

$y'' + 4y' + 4y = 10\cos 3t$

1

b) Find the position of the mass at any time t. (3pts)

given

$m = 1kg$

$l = 4kg/sec$

$k = 4\,N/m$

$F(t) = 10\cos 3t$

$x(0) = 0$

$x'(0) = 2$

for homogeneous equation.

$y'' + 4y' + 4y = 0$

$\lambda^2 + 4\lambda + 4 = 0$

$\lambda^2 + 2\lambda + 2\lambda + 4$

$\lambda(\lambda + 2) + 2(\lambda + 2)$

$(\lambda + 2) = 0$

$\lambda = -2$

$y_1 = e^{-2x}$

3

$y_h(x) = y_1 c_1 + y_2 c_2$

$y_h(x) = c_1 e^{-2x}$

* $r(x) = 10\cos 3t$ to find

$r(x) = A\cos 3t + B\sin 3t$

$r'(x) = -3A\sin 3t + 3B\cos 3t$

$r''(x) = -9A\cos 3t - 9B\sin 3t$

Good luck

$y'' + 4y' + 4y = 10\cos 3t$

~~$9A\cos 3t - 9B\sin 3t - 12A\sin 3t + 12B\cos 3t + \dots$~~

$-9A\cos 3t - 9B\sin 3t - 12A\sin 3t + 12B\cos 3t + 4A\cos 3t + 4B\sin 3t = 10\cos 3t$

$-5A + 12B = 10$

$-5B - 12A = 0$

$-5B = 12A$

$B = -\frac{12}{5}A$

from $-5A + 12B = 10$

$-5A + 12\left(-\frac{12}{5}A\right) = 10$

$-5A - \frac{144}{5}A = 10$

$\frac{-25A - 144A}{5} = 10$

$\frac{-169A}{5} = 10$

$A = \frac{-50}{169}$

then $B = -\frac{12}{5}A$

$B = \frac{-12}{5} \times \left(\frac{-50}{169}\right)$

$B = \frac{120}{169}$

$y_p(x) = A\cos 3t + B\sin 3t$

$y_p(x) = \frac{-50}{169}\cos 3t + \frac{120}{169}\sin 3t$

Then general solution

$y(x) = y_h(x) + y_p(x)$

~~$y(x) = c_1 e^{-2x} \dots$~~

$\therefore y(x) = c_1 e^{-2x} - \frac{50}{169}\cos 3t + \frac{120}{169}\sin 3t$ (Ans)

$y'' + 4y' + 4y = 10\cos 3t$

~~$9A\cos 3t - 9B\sin 3t - 12A\sin 3t + 12B\cos 3t +$~~

$A\cos 3t - 9B\sin 3t - 12A\sin 3t + 12B\cos 3t + 4A\cos 3t + 4B\sin 3t = 10\cos 3t$

$-5A + 12B = 10$

$-5B - 12A = 0$

$-5B = 12A$

$B = -\frac{12}{5}A$

from $-5A + 12B = 10$

$-5A + 12\left(-\frac{12}{5}A\right) = 10$

$-5A - \frac{144}{5}A = 10$

$\frac{-25A - 144A}{5} = 10$

$\frac{-169A}{5} = 10$

$A = \frac{-50}{169}$

then $B = -\frac{12}{5}A$

$B = \frac{-12}{5} \times \left(\frac{-50}{169}\right)$

$B = \frac{120}{169}$

$y_p(x) = A\cos 3t + B\sin 3t$

$y_p(x) = \frac{-50}{169}\cos 3t + \frac{120}{169}\sin 3t$

Then general solution

$y(x) = y_h(x) + y_p(x)$

~~$y(x) = c_1 e^{-2x} t$~~

$\therefore \; y(x) = c_1 e^{-2x} - \frac{50}{169}\cos 3t + \frac{120}{169}\sin 3t$ (Ans)

5. Find the general solution of the system of differential equation (5 pts)

$$\begin{cases} x' = x+2y \\ y' = 4x+3y \end{cases}$$

$= x' - x - 2y = 0 \quad = \begin{cases} (D-1)x - 2y \\ (D-3)y - 4x \end{cases}$

$y' - 3y - 4x = 0$

$\begin{bmatrix} D-1 & -2 \\ -4 & D-3 \end{bmatrix} \Rightarrow (D-1)(D-3) - 8 = 0$

$D^2 - 3D - D + 4 - 8 = 0$

$D^2 - 4D - 4 = 0$

Two the same as

$x(t) = x'' - 4x' - 4x = 0$

$y(t) = y'' - 4y' - 4y = 0$

2

$(D-1)x - 2y$

$(D-3)y - 4x$

$\begin{matrix} D-1 & -2 \\ -4 & D-3 \end{matrix}$

write the characteristic equation

$\lambda^2 - 4\lambda - 4 = 0$

$\lambda = \frac{4 \pm \sqrt{16+16}}{2} \Rightarrow \frac{4 \pm \sqrt{32}}{2} \Rightarrow \frac{4 \pm 4\sqrt{2}}{2}$

$\lambda_1 = 2 \pm 2\sqrt{2}$ $\quad y' + y(x) = \cdot$

General solution

$\lambda_2 = 0$

$x(t) = c_1 x_1 + c_2 x_2$

$\therefore x(t) = c_1 e^{(2+2\sqrt{2})x} + c_2 e^{(0)x} \Rightarrow c_1 e^{(2+2\sqrt{2})x} + c_2$

$\therefore y(t) = c_1 e^{(2+2\sqrt{2})x} + c_2 e^{(0)x} \Rightarrow c_1 e^{(2+2\sqrt{2})x} + c_2$

$x'(t) = 2 + \frac{1}{\sqrt{2}x} \cdot c_1 e^{(2+2\sqrt{2})x}$

Inserting in $x' - x - 2y = 0$

$2 + \frac{1}{\sqrt{2}x} \cdot c_1 e^{(2+2\sqrt{2})x} - (c_1 e^{(2+2\sqrt{2})x} + c_2) - 2(c_1 e^{(2+2\sqrt{2})x} + c_2)$

$c_1 e^{(2+2\sqrt{2})x} - 3c_1 e^{(2+2\sqrt{2})x} - 3c_2$ 8 (Ans)

Good luck

$\therefore x(t) = 2 + \frac{1}{\sqrt{2}x} \cdot c_1 e$

$x' = x - 2y = 0$

$y' - 4x + 2y$

-7.5

Applied Math. III Mid-exam

ASTU (15.5 Nov. 25 08, 2014

ADAMA SCIENCE AND TECHNOLOGY UNIVERSITY
SCHOOL OF NATURAL SCIENCE
DEPARTMENT OF MATHEMATICS
Applied Mathematics III
Mid Examination

- *Work is required for full credit, and may earn partial credit. (for work-out problems)*
- **Use the back side if necessary**
- ***Mobile phone is not permitted.***
- ***This is the first of five (5) pages.***

PART I: Short Answer questions. Write the most simplified answer on the space provided. (10 %)

1. Classify the following differential equations as linear or non-linear and state their order.

	Linear/Nonlinear	Order
$y' + y = \sin y$	Non-Linear	1st
$x^2y'' + 3xy' - y\cos x = 0$	Linear	2nd

2. The differential equation $2xy^3 - 3y - (3x + ax^2y^2 - 2ay)y' = 0$ is exact. Then the value(s) of a is α

3. Let $y_1(x) = 1$ is a solution of $(1 - x^2)y'' + 2xy' = 0$. Then the second solution $y_2(x) =$ $x - \frac{x^3}{3}$

4. Given the initial value problem $y'' + y' - 2y = 0$ $y(0) = \beta,\ y'(0) = 1$. What value(s) of β will make $\lim_{x\to\infty} y(x) = 0$. Answer $\beta =$ $\cancel{\ \ }$

5. The integrating factor for $y - xy' = 0$ are $\frac{1}{x^2}$ and $\frac{1}{y^2}$

1 Mid-Exam

-26-

$\frac{2x}{1-x^2}$

(2010)

$y'' + f(x)y' + Q(x)y = g(x)$

ASTU Nov. 25 08, 2014

Part II: Work out problems. Show all the necessary steps clearly and neatly on the space provided. Unreadable steps and answers worth no point.

** *For problem 1 and 3 you have to choose one problems to do from the given alternatives.* (20 %)-Each 5 pt

1. (a*) A 500 gallon tank initially contains 50 gallons of brine solution in which 28 *lbs* of salt have been dissolved. Beginning at time zero, brine containing 2 *lbs/gal* of salt is added at a rate of 3 *gal/min* and the mixture is poured out of the tank at the rate of 2 *gal/min*. How much salt is in the tank when it contains 100 gallons of brine.

   (b*) A container of hot liquid is placed in a room of temperature 19^0c and in 8 minutes the liquid cools from 83^0c to 51^0c. How long does it takes for the liquid to cool from 27^0c to 25^0c?

Soln

Q(t) = (flow rate)in Concentration - (flow rate)out Concentration

Concentration = 2 lbs/gal

flow rate in = 3 gal/min

Concentration out = $\frac{A(t)}{500+t}$

flow rate out = 2 gal/min

$A(t) = (3)(2) - 2\left(\frac{A(t)}{500+t}\right)$

$A'(t) = 6 - 2\left(\frac{A(t)}{500+t}\right)$

$A'(t) = 6 - 2\left(\frac{100}{500+t}\right)$

$A'(t) = 6 - \frac{200}{500+t}$

$\frac{dA(t)}{dt} = 6 - \frac{200}{500+t}$

$dA(t) = \int 6 - \frac{200}{500+t}\, dt$

$A(t) = 6t - 200 \ln|500+t|$

2. Solve

2

$$\frac{dx}{dt} - 3y + 16\cos t = 0$$
$$3x + \frac{dy}{dt} = 0$$

subject to $x(0) = 2$, $y(0) = 0$

Sol:

$3y = \frac{dx}{dt} + 16\cos t$

$y = \frac{1}{3}\frac{dx}{dt} + \frac{16}{3}\cos t$

$\boxed{\frac{dy}{dt} = \frac{1}{3}\frac{d^2x}{dt^2} + \frac{16}{3}(-\sin t)}$ ... Eq 1

Substitute Eq 1 in

$3x + \frac{dy}{dt} = 0 \Rightarrow 3x + \frac{1}{3}\frac{d^2x}{dt^2} - \frac{16}{3}\sin t = 0$

$3x + \frac{1}{3}x'' - \frac{16}{3}\sin t = 0$

$\Rightarrow \frac{1}{3}x'' + 3x = \frac{16}{3}\sin t$

$x'' + 9x = 16\sin t$

$\boxed{x = x_h + x_p}$

$x_h \Rightarrow x'' + 9x = 0$

$\lambda^2 + 9 = 0$

$\lambda^2 = -9 \Rightarrow \lambda = \pm\sqrt{-9}$

$\lambda_1 = 3i$ or $\lambda_2 = -3i$

$x_h = e^{x}(\cos \omega x - \cos \omega x)$

$= e^{3x}(\cos x - \cos -x)$

$x_p \Rightarrow r_n \sin n \Rightarrow r_n \cos n$

$x = x_h + x_p \Rightarrow x = e^{3x}(\cos x - \cos x) + 16\cos t = x(t)$

~~$x(t)$~~ $x(0) = 16\cos t = 2$

$\cos t = \frac{1}{8}$

$t = \cos^{-1}\left(\frac{1}{8}\right)$

3. (a*) A 4 ft spring measures 8 ft long after a mass weighting 8 lb is attached to it. The medium through which the mass offers a damping force numerically equal to $\sqrt{2}$ times the instantaneous velocity. (Take $g = 32\ ft/sec^2$) Find

(a) the equation of motion if the mass is initially released from equilibrium position with down ward velocity of 5 ft/sec.

(b) the time at which the mass attains its extreme displacement from the equilibrium position.

(c) what is the position of the mass at this instant?

(b*) An RLC series circuit connected in series with a resistance of $R = 6\Omega$, capacitor $c = 0.02\ F$ and inductance of $L = 0.1\ H$ has applied voltage $E(t) = 6\ V$. Assuming no initial charge on capacitor and no initial current in the circuit, then find the current and the charge at any time t.

Sol^n

b. $q'' + \frac{R}{L}q' + \frac{1}{LC}q = \frac{E(t)}{L}$, $R = 6\Omega$, $C = 0.02F$, $L = 0.1H$, $E(t) = 6V$

$q(0) = 0$

$q(t) = ?$

$q'' + \frac{6}{0.1}q' + \frac{1}{(0.1)(0.02)}q = \frac{6}{0.1}$

$\Rightarrow q'' + 60q' + 500q = 60$

It is linear eq^n

$q(t) = e^{-\int p(t)dt}\left[\int e^{\int p(t)dt} R(x)\,dx + c\right]$

$= e^{-\int 60dt}\left[\int e^{\int 60dt} \cdot 60\,dt + c\right]$

$= e^{-60t}\left[\int e^{60t} \cdot 60\,dt + c\right]$

$= e^{-60t}\left[60 \cdot \frac{1}{60} e^{60t} + c\right]$

$= e^{-60t}\left[e^{60t} + c\right]$

$q(t) = ce^{-60t}$ $\Rightarrow$ $I(t) = \frac{dq(t)}{dt}$

$I(t) = \frac{d\,ce^{-60t}}{dt}$

$I(t) = -60ce^{-60t}$

4. Solve the following differential equation

(a) $(x-y)dx-(y-x)dy=0$

$M = x+y$ $\qquad N = -y+x$

$\frac{dM}{dy} = 1 \qquad \frac{dN}{dx} = 1$ ... (Exact)

$U(x,y) = \int M\,dx + k(y)$

$= \int (x+y)\,dx + k(y)$

$U(x,y) = \frac{x^2}{2} + yx + k(y)$

$\frac{du}{dy} = x + \frac{dk(y)}{dy} = -y+x = N$ .... differentiating interms of y

$= \frac{dk(y)}{dy} = -y \Rightarrow \int dk(y) = \int -y\,dy$

$k(y) = -\frac{y^2}{2}$

$\therefore U(x,y) = \frac{x^2}{2} + yx - \frac{y^2}{2}$

(b) $x^2y'' - 5xy' + 9y = 0$

$\Rightarrow x^2y'' - 5xy' + 9y = 0$

$m^2 + (a-1)m + b = 0$

$m = \frac{-(a-1) \pm \sqrt{(a-1)^2 - 4b}}{2} = \frac{-(-5-1) \pm \sqrt{(-5-1)^2 - (4)(9)}}{2}$

$m = \frac{6 \pm \sqrt{36-36}}{2} \Rightarrow m = \frac{6}{2}$

$m = 3$

Since it has one root

$y = (c_1 + c_2 \ln x)x^m$

$y = (c_1 + c_2 \ln x)x^3$

-30-

Alternative Form of $x(t)$ In a manner identical to the procedure used on page 218, we can write any solution

$x(t) = e^{-\lambda t}(c_1 \cos\sqrt{\omega^2-\lambda^2}\,t + c_2 \sin\sqrt{\omega^2-\lambda^2}\,t)$ (23)

in the alternative form

... is determined from the

2. The general solution of $\frac{dy}{dx}+\left(1-\frac{2}{x}\right)y=3x^2e^{-x}$ is $(1-\frac{2}{x})y$ ____________

$I = \int^{P(x)}$ $\frac{3x^3+}{e^x}$

3. The general solution of the non linear DE $xy''+y'=e^{-x}$ is ____________

4. If $y_1(x)=x^2$ is the first solution of the differential equation $x^2y''-2xy'+2y=0$, then the second linearly independent solution is $y_2(x)$ $= x^2$ ____________

5. If $y(x)=(x+3)e^{2x}$ is a solution of the initial value problem $\frac{dy}{dx}=ay+e^{2x}$, $y(0)=b$ then, the values of $a=$ ____________, $b=$ 3 ____________

$y' = ay + e^{2x}$

6. A linear homogenous second order ordinary differential equation with constant coefficients whose linearly independent solutions are $e^{-2\pi x}$ and $e^{2\pi x}$ is $y = c_1e^{-2\pi x} + c_2e^{2\pi x}$

7. The integrating factor(s) for $(2-xy^2)dx+2x^2ydy=0$ is (are) ____________

8. Let $f(x)=\begin{cases}-x; & -\pi\le x<0\\ x; & 0\le x\le\pi\end{cases}$ such that $f(x+2\pi)=f(x)$ for all x. Then, the Fourier coefficients are:

$a_0=$ ~~$\frac{1}{\pi}$~~ ~~2π~~, $a_n= \frac{1}{\pi}\int_{-\pi}^{0} -x\cos nx + \frac{1}{\pi}\int_0^{\pi} x\cos nx$, $b_n= \frac{1}{\pi}\int_{-\pi}^{0} -x\sin nx + \frac{1}{\pi}$

PART II: WORKOUT PROBLEMS. Show all the necessary steps clearly and neatly on the space provided. (20%)

1.5

1) Solve the following differential equations

a) $x^2\frac{dy}{dx}=3(x^2+y^2)\arctan\left(\frac{y}{x}\right)+xy$

$x^2\frac{dy}{dx} = 3(x^2+y^2)\tan^{-1}(\frac{y}{x})+xy$

$\frac{x^2y'}{x^2} = \frac{3(x^2+y^2)}{x^2}\tan^{-1}(\frac{y}{x}) + \frac{xy}{x^2}$

$y' = \frac{3(x^2+y^2)}{x^2}\tan^{-1}(\frac{y}{x}) + \frac{xy}{x^2}$

$y' = 3(1+\frac{y^2}{x^2})\tan^{-1}(\frac{y}{x}) + \frac{y}{x}$

$\boxed{\text{let } u = \frac{y}{x}}$ $\boxed{y = ux}$

0.5

$y' = u'x + u$

$y' = 3(1+u^2)\tan^{-1}(u) + u$

$u'x + u = 3(1+u^2)\tan^{-1}(u) + u$

$u = 3(1+u^2)\tan^{-1}(u) + u$

sep. arrouble

$u'x = 3(1+u^2)\tan^{-1}(u) + u$

$u'x = 3(1+u^2)\tan^{-1}(u)$

$x\frac{du}{dx} = 3(1+u^2)\tan^{-1}(u)$

$\frac{du}{3(1+u^2)\tan^{-1}(u)} = \frac{dx}{x}$

$\int \frac{du}{3(1+u^2)\tan^{-1}(u)} =$

$\int \frac{du}{3(1+u^2)\tan^{-1}(u)}$

~~let~~

let $v = 3(1+u^2)$

$\int \frac{dv}{v} = \int \frac{dx}{x}$

$\ln v = \ln x + c$

$3\ln(1+u^2)\tan^{-1}(u)$

$\ln(1+u^2)$ ~~$\tan^{-1}(u)$~~

b) $x^2y''+5xy'+4y=0$

Euler-Cauchy equation

$A=5$ $\boxed{x=e^t}$

$B=4$

Characteristic:

$h^2+(5-1)h+4$

$h^2+4h+4=0$

$h^2+4h+4=0$

$h^2+2h+2h+4=0$

$h(h+2)+2(h+2)=0$

$h_{1,2}=-2$

$\boxed{\text{Repeated root}}$

0.5

0.5

2

c) $\frac{dy}{dx}+\frac{1}{x}y=x^2y^4$ Ber

$\downarrow$

Bernoulli equation

$y'+\frac{1}{x}y=x^2y^4$

$\boxed{n=4}$

$y'+\frac{1}{x}\ \ \ \ x^2y^4$

1.5

$y(t)=$

$y(t)=$

$y(t)=$

$y(t)$

}3

$x(t) = e^{-\lambda t}(c_1 \cos\sqrt{\omega^2 - \lambda^2}\,t + c_2 \sin$

in the alternative form

$\sin(\sqrt{\omega^2 - \lambda^2}\,t$

p(a) =

2

m(x,y)dx $6x^5 - xy$

n(x,y) dy $= x^2y^2 - y^2$

d) $(6x^5 - xy)dx + (xy^2 - x^2)dy = 0$

$x(6x^4 - y) + y(y^2 - x)$

$m(x,y)dx = 6x^5 - xy$ $n(x,y)dy = xy^2 - x^2$

$\frac{dm}{dy} = -x$ $\frac{dn}{dx} = y^2 - 2x$

It is not an exact equation

$f(x) = \frac{M_y - N_x}{N} = \frac{-x - xy}{2xy - 2x}$

$f(x) = \frac{M_y - N_x}{N} = \frac{-x - (y^2 - 2x)}{xy^2 - y^2}$

$f(y) = \frac{N_x - M_y}{M} = \frac{(y^2 - 2x) - (-x)}{6x^5 - xy} =$

$f(y) = \frac{y^2 - 2x + x}{6x^5 - xy} = \boxed{\frac{y^2 - x}{6x^5 - xy}}$

2. Solve the system of differential equation

$$\begin{cases} \frac{dx}{dt} + y = t, \\ x + \frac{dy}{dt} = 0. \end{cases}$$

Alternative Form of $x(t)$ In a manner identical to the ... on page 218, we can write any solution

$$x(t) = e^{-\lambda t}(c_1 \cos\sqrt{\omega^2 - \lambda^2}t + c_2 \sin\sqrt{\omega^2 - \lambda^2}t) \quad (23)$$

in the alternative form

$x(0) = -2$

3. A tank initially contains 200 liters of water with 100 grams of salt in solution. Water containing salt with concentration of 1gm/liter is poured in at the rate of 3 L/min. The well-stirred water is allowed to pour out the tank at a rate of 2 L/min.
   a) Formulate initial value problem describing the phenomena.
   b) Find the amount of salt in the tank at any time $t \geq 0$.
   c) When is the tank empty?

Given — Required

$V_0 = 200L$ — $A(t) = ?$

$A(0) = 100$ gram — $V_0 =$

$R_{in} = 3 L/min$

$R_{out} = 2 l/min$

$V_0 = 200L$

$A'(t) = 3 L/min \cdot 1g/L - 2 L/min \cdot \frac{A(t)}{V_0 + t}$

$A'(t) = 3 - \frac{2A(t)}{100L + t}$

$A'(t) + \frac{2A(t)}{200L + t} = 3$

$\boxed{A'(t) + \frac{2A(t)}{200+t} = 3}$ = (a)

$A'(t) + \frac{2A(t)}{200+t} = 3$

$P(x) = \frac{2}{200+t}$

$e^{\int P(x)dx} = e^{2\ln(200+t)} = (200+t)^2$

$(200+t)^2 A'(t) + \frac{2A(t)}{(200+t)}(200+t)^2 = 3(200+t)^2$

$(200+t)^2 A'(t) + \frac{2(200+t)^2 A(t)}{200+t} = 3(200+t)^2$

$(200+t)^2 A'(t) = 3(200+t)^2$

$\frac{(200+t)^3}{3} A(t) = \frac{3(200+t)^3}{3} + C$

$\frac{(200+t)^3}{3} A(t) = (200+t)^3 + C$

$\frac{(200+t)^3}{3} A(t) = (200+t)^3$

$A(t) = \frac{3(200+t)^3}{(200+t)^3}$

$\boxed{A(t) = 3 + \frac{3C}{(200+t)^3}}$

General solution

(b) $\boxed{A(t) = \frac{3C}{(200+t)^3} + 3}$

(c) $= A(t) = 0$

$\frac{3C}{(200+t)^3} + 3 = 0$

$\frac{3C}{(200+t)^3} = -3$

$A(0) = 100$ g/m

$\frac{3C}{(200)^3} + 3 = 100$

$\frac{3C}{(200)^3} = 97$

$\frac{3C}{3} = \frac{97 \times 80}{3}$

$\boxed{C = \frac{97 \times 80}{3}}$

$A(t) = 3 + \ldots$

III Mid Exam

4. A spring with a 3kg mass is held stretched 0.6m beyond its natural length by a force of 20N. If the spring begins at its equilibrium position but a push gives it an initial veloci[ty] of 1.2m/s. Find

a) The position of the mass after time t
b) Amplitude, frequency and period

$ma = -ky$

$ky = 20N$

$\frac{ky}{y} = \frac{20N}{y}$

$k = \frac{20N}{0.6}$

$k = \frac{200N}{6}$

$k = 33$

Given

$M = 3kg$

$y = 0.6\,m$

$ky = 20N$

$y'(0) = -1.2\,m/s$

$y(0) = 0$

$y'(0) = -1.2\,m/s$

$My'' = -ky$

$\frac{my''}{m} = \frac{ky}{m}$

$y'' + \frac{k}{m}y = 0$

$y'' + \frac{33y}{3} = 0$

$y'' + 11y = 0$

Auxiliary: $h^2 + 11 = 0$

$h = \pm\sqrt{11}\,i$

$y(t) = c_1\cos\sqrt{11}\,t + c_2\sin\sqrt{11}\,t$

(a) $y(t) = c_1\cos\sqrt{11}\,t + c_2\sin\sqrt{11}\,t$

let $y(0) = c_1\cos\sqrt{0} + c_2\sin\sqrt{10}$

$c_1 = 0$

$y'(x) = -c_1\sqrt{11}\sin\sqrt{11}x + c_2\cos$

$\sqrt{11}\,c_2 = -1.2$

$c_2 = \frac{-1.2}{\sqrt{11}}$

$c_1 = 0$

$c_2 = \frac{-1.2}{\sqrt{11}}$

$A = \sqrt{(c_1)^2 + (c_2)^2}$

$A = \sqrt{(0)^2 + \ldots}$

$A = \frac{1.2}{\sqrt{11}}$

b =